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Fluctuation-Dissipation Theorem and the Dynamical Renormalization Group

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The relation between a recently introduced dynamical real-space renormalization group and the fluctuation-dissipation theorem is discussed. An apparent incompatibility is pointed out and resolved.

KEY WORDS: Renormalization group; fractals; Langevin dynamics.

1. INTRODUCTION

Over the years, Langevin dynamics has been proven to be a successful theoretical approach to nonequilibrium problems such as critical dynamics,[1] growth processes,[2] and interface dynamics.[3] In a previous paper[4] an exact dynamical real-space renormalization group analysis of Langevin dynamics derivable from a Gaussian field theory was presented. The method has the same range of applicability as the static counterpart, namely, one-dimensional lattices and a whole set of hierarchical or self-similar structures. Apart from the methodological interest of the scheme per se, the results of the method are expected to be of practical interest as well, in view of the realization of growth processes on electrochemical electrodes.[5] Unlike other real-space approximate schemes,[6] one important

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feature of this approach is that it contains the static analog as a particular case. This naturally raises the question of the compatibility of the scheme with the standard fluctuation-dissipation theorem[7] which relates the mobility (assumed unity in the rest of the paper) to variance of the noise which drives the system. In this paper, we will address this issue within the framework of an exactly solvable model. This model, although simple, is important since it is the zeroth-order one in any perturbative expansion.

Consider a system described by a Gaussian field theory in one dimension with Hamiltonian (or action) $H(\{\varphi\})$ given by

$$H(\{\varphi\}) = \frac{1}{2} \sum_{x, y} \varphi_x A_{x, y} \varphi_y \tag{1}$$

with the field variables φ_x defined on the sites x of a lattice. As it is well known,[5] the implementation of an *exact* static real-space renormalization group analysis is possible only in the case when the matrix $A_{x, y}$ has a nearest neigbor restriction:

$$A_{x, y} = a_x \delta_{x, y} - \delta_{|x-y|, 1} \tag{2}$$

The renormalization procedure has three basic steps:

1. The set E of lattice points is divided into E_s (sites which survive the decimation) and E_d (sites which are decimated), so that their union is E and their intersection is null. The degrees of freedom corresponding to E_d are then eliminated either by integration over the corresponding field variables or by elimination of the corresponding equations of motion (in the dynamical case).
2. The surviving fields are rescaled so that the new Hamiltonian has the form as the original one.
3. Lengths are scaled in such a way that the original lattice constant is recovered.

As result of this procedure, a mapping between the original and the new set of parameters is obtained.

Let us now recall the results of the same procedure in dynamics.[4] The simplest Langevin dynamics can be constructed as

$$\frac{\partial \varphi_x(t)}{\partial t} = -\frac{\delta H(\{\varphi\})}{\delta \varphi_x} + \eta_x(t) = -\sum_y A_{x, y} \varphi_y(t) + \eta_x(t) \tag{3}$$

[5] There is a huge literature on this topic; see, e.g., ref. 8.

where the stochastic noise $\eta_x(t)$ is chosen from a Gaussian distribution which has a zero average and variance

$$\langle \eta_x(t_1)\, \eta_y(t_2) \rangle = 2D_{x,y}\, \delta(t_1 - t_2) \tag{4}$$

The renormalization scheme works along the same lines as in the static case, but with the important difference that the renormalization of the noise also has to be taken into account. It was shown in ref. 4 that a necessary and sufficient condition for the implementation of the renormalization procedure is that the matrix $D_{x,y}$ has the form

$$D_0 \delta_{x,y} + D_1 \delta_{|x-y|,1} \tag{5}$$

Since both D_0 and D_1 are different from zero under renormalization, the minimum parameter space for the fluctuation matrix D is given by $\{D_0, D_1\}$, implying nearest neighbor correlation of the noise. We stress once again that both the static and the dynamic schemes we are dealing with are *exact*, i.e., closed in the parameter space.

We now turn to the connection with the fluctuation-dissipation theorem. The Fokker–Planck equation associated with the Langevin dynamics (3) with a noise whose variance is (4) is[7]

$$\frac{\partial}{\partial t} P(\{\varphi\}, t) = \sum_x \frac{\delta}{\delta \varphi_x} \left[P(\{\varphi\}), t) \frac{\delta}{\delta \varphi_x} H(\{\varphi\}) + \sum_y D_{x,y} \frac{\delta}{\delta \varphi_y} P(\{\varphi\}, t) \right] \tag{6}$$

It is then easy to show from (6) that, if we denote by $P_{\mathrm{eq}}(\{\varphi\})$ the equilibrium probability distribution obtained in the $t \to \infty$ limit, a necessary condition for ($\mathcal{N}^{-1}$ is a normalization factor)

$$P_{\mathrm{eq}}(\{\varphi\}) = \mathcal{N}^{-1} e^{-H(\{\varphi\})} \tag{7}$$

to be satisfied is that the matrix $D_{x,y}$ has a *diagonal* form, so that in the case of Eq. (5) $D_1 = 0$. Indeed (3)–(5) with $D_1 = 0$ lead to the equilibrium distribution given by (7). However, the dynamical renormalization group *violates*, in general, the fluctuation-dissipation theorem, since one has to start with a matrix with a nondiagonal form and the equilibrium distribution of (7) is not obtained.

We now argue that this inconsistency is only apparent. We will show that the $D_1 \neq 0$ case will lead to an equilibrium distribution of a more general Gaussian Hamiltonian. However, we will explicitly demonstrate that equilibrium correlation functions corresponding to the two Hamiltonians are characterized by the same spatial decay.

Let us consider a more general Gaussian Hamiltonian:

$$\tilde{H}(\{\varphi\}) = \tfrac{1}{2} \sum_{x,y} \varphi_x \tilde{A}_{x,y} \varphi_y \tag{8}$$

where the symmetrix matrix $\tilde{A}_{x,y}$ is *not* necessarily restricted to nearest neighbors. On again using Eq. (6), it can be shown readily that a Langevin equation of the more general form

$$\frac{\partial \varphi_x(t)}{\partial t} = -\sum_y D_{x,y} \frac{\delta \tilde{H}(\{\varphi\})}{\delta \varphi_y} + \eta_x(t) \tag{9}$$

leads to the equilibrium distribution

$$P_{\text{eq}}(\{\varphi\}) = \mathcal{N}^{-1} \exp(-\tilde{H}(\{\varphi\})) \tag{10}$$

Thus if one starts with the model defined by Eqs. (1)–(4) with $D_0 = 1$ and $D_1 = 0$, so that the equilibrium distribution is given by Eq. (7), then the dynamical renormalization group (DRG) leaves the Langevin equation of the form (3) with A and D given by Eqs. (2) and (4) respectively, with $D_1 \neq 0$ after one RG step.

From Eqs. (8)–(10), the equilibrium distribution is given by (10), with $\tilde{H}$ given by (8) and

$$\tilde{A} = D^{-1} A \tag{11}$$

$\tilde{A}$ is symmetric since both D and A depend only on the difference $|x-y|$. Thus initially $\tilde{A} = A$ and then the DRG flow for $\tilde{H}$ occurs in a wider parameter space than the original one given by Eq. (2) [i.e., $\tilde{A}$ no longer has the form (2)]. On the contrary, a static RG would leave H of the form (1)–(2). However, as we show below, these two Hamiltonians are *equivalent* in the sense that the corresponding correlation functions have the same leading behavior. It is noteworthy that a *static* Gaussian model with interactions defined by $\tilde{A}$ appearing in the Hamiltonian (8) *not* restricted to short range can be exactly renormalized through the dynamics, provided that the matrix $D \cdot \tilde{A}$ *is* restricted to nearest neighbor interactions!

As an explicitly solvable example, let us consider the simplest case of a one-dimensional lattice with $a_x = a$. The static recursions are:[8]

$$a' = a^2 - 2 \tag{12}$$

The corresponding dynamics leads to the following recursions[4]:

$$\alpha'(\omega') = \alpha^2(\omega) - 2 \tag{13a}$$

$$D'_0 = \tfrac{3}{4} D_0 + D_1 + o(\omega) \tag{13b}$$

$$D'_1 = \tfrac{1}{8} D_0 + \tfrac{1}{2} D_1 + o(\omega) \tag{13c}$$

where $\alpha(\omega) \equiv a - i\omega$ and ω is the Fourier variable conjugate to time.

From Eq. (13a) it is apparent that the static case is recovered in the limit $\omega \to 0$ (i.e., $t \to \infty$), that is, the statics is included in the dynamics, as we mentioned already.

Let us now discuss the two-point correlation function given by

$$G_{x,y} = \langle \varphi_x \varphi_y \rangle = \frac{1}{Z} \int \mathscr{D}\varphi \, e^{-H(\{\varphi\})} \varphi_x \varphi_y = (A^{-1})_{x,y} \tag{14}$$

where Z is the partition function and $\mathscr{D}\varphi \equiv \prod d\varphi_x$. A similar expression holds for $\tilde{G}_{x,y}$ corresponding to the Hamiltonian $\tilde{H}(\{\varphi\})$.

Due to the particular form of A and D and using Eq. (11), one can easily see that the $G_{x,y}$ and $\tilde{G}_{x,y}$ are given by (with the lattice constant set to unity)

$$G_{x,y} = \int_{-\pi}^{+\pi} \frac{dq}{2\pi} e^{iq(x-y)} \frac{1}{a - 2\cos q} \tag{15}$$

and

$$\tilde{G}_{x,y} = \int_{-\pi}^{+\pi} \frac{dq}{2\pi} e^{iq(x-y)} \frac{D_0 + 2D_1 \cos q}{a - 2\cos q} \tag{16}$$

The integrals can be carried out exactly and in the $a > 2$ case, where the correlation functions are real, they yield the same coarse-grained behavior i.e.,

$$G_{x,y} = \frac{1}{(a^2 - 4)^{1/2}} e^{-\lambda |x-y|} \tag{17}$$

(when $|x - y| > 1$) and

$$\tilde{G}_{x,y} = \frac{D_0 + D_1 a}{(a^2 - 4)^{1/2}} e^{-\lambda |x-y|} \tag{18}$$

where we have defined

$$\lambda = \left| \ln \left\{ \frac{a}{2} \left[1 - \left(1 - \frac{4}{a^2} \right)^{1/2} \right] \right\} \right| \tag{19}$$

This shows that the two Hamiltonians H and $\tilde{H}$ are indeed equivalent, in the sense that the corresponding correlation functions decay similarly.

Conversely, one can show that $\tilde{A}$ may be written in the form (11) with A and D given by (2) and (5), respectively, provided that, at leading order, it is of the form

$$\tilde{A}_{x,y} = \tilde{a}\delta_{x,y} + \tilde{b} e^{-\mu |x-y|} \tag{20}$$

In summary, we have discussed the issue of the relationship between the dynamical renormalization group and the fluctuation-dissipation theorem, which was prompted by our previous analysis. We showed that although at first glance there is a violation of the fluctuation-dissipation theorem, a more careful analysis restores its validity.

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